Generalizing nowhere dense graph classes

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CSGT

Overview

Nowhere dense classes

Definition

A *monotone graph class* is closed under taking subgraphs (or equivalently, is determined by forbidden subgraphs).

- For many questions, a monotone class is as complicated as the class of all graphs iff it is *somewhere dense*.
- Example: the dividing line for first-order model checking.
- Containing arbitrarily large cliques is bad.

Definition

A graph class $\mathcal{C}$ is *somewhere dense* if there is some k such that $\mathcal{C}$ contains arbitrarily large k-subdivided cliques (or complete bipartite graphs).

Generalization goals

- First goal: monotone classes of relational structures (partially ordered hypergraphs, . . .)
- Second goal: hereditary classes (graphs or relational structures)

Definition

A *hereditary graph class* is closed under taking *induced* subgraphs (or equivalently, is determined by forbidden *induced* subgraphs).

- Example: the class of all cliques is a very simple hereditary class, but is somewhere dense.

(Monadic) Stability and NIP

- *Stability* and *NIP* are classic dividing lines in model theory.

Definition

A class $\mathcal{C}$ is *NIP* if it does not encode all finite bipartite graphs.
A class $\mathcal{C}$ is *stable* if it does not encode arbitrarily large half-graphs.

- *NIP* is equivalent to certain natural set systems having bounded VC-dimension.

Definition

A class $\mathcal{C}$ is *monadically stable/NIP* if it remains stable/NIP under arbitrary vertex-colorings.

Relation to nowhere density

Theorem (Podewski-Ziegler '78, Adler-Adler '14)

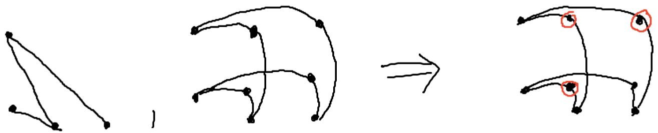
A monotone graph class $\mathcal{C}$ is nowhere dense $\iff$ $\mathcal{C}$ is (monadically) stable $\iff$ $\mathcal{C}$ is (monadically) NIP.

- Provides a route for generalizing nowhere density.

Theorem (BGOSTT '21)

Let $\mathcal{C}$ be a hereditary class of ordered graphs. Then $\mathcal{C}$ is (monadically) NIP $\iff$ $\mathcal{C}$ has bounded twin-width.

- Easy: For monotone $\mathcal{C}$, somewhere dense implies not monadically NIP.

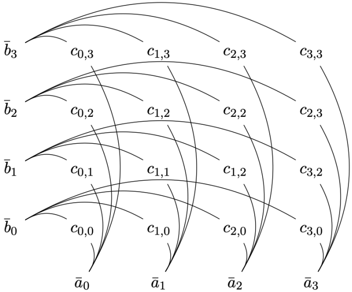


Forbidden configuration: Grids

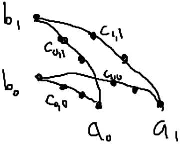
- With Laskowski, we give several characterizations of monadic NIP.

Lemma (Shelah ’86, B.-Laskowski ’21)

If $\mathcal{C}$ is not monadically NIP, then $\mathcal{C}$ codes arbitrarily large grids (on tuples).

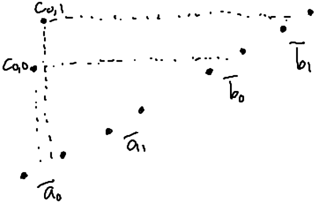


Examples of grids



$\phi(x,y,z) := \exists z' \, (x R z \wedge z R z' \wedge z' R y)$

Figure: 2-subdivided $K_{n,n}$. Note ϕ is *positive existential*.



$\phi(x_0 x_1, y_0 y_1, z) :=$

$x_0 <_1 z <_1 x_1 \wedge y_0 <_2 z <_2 y_1$

Figure: Permutations

Collapse

Lemma (B.-Laskowski '22+)

If $\mathcal{C}$ is not monadically NIP, then the grid can be defined by a formula with low quantifier-complexity (a boolean combination of existentials).

Theorem (B.-Laskowski '22+)

Let $\mathcal{C}$ be a hereditary class of relational structures.
$\mathcal{C}$ is NIP $\iff$ $\mathcal{C}$ is monadically NIP.
$\mathcal{C}$ is stable $\iff$ $\mathcal{C}$ is monadically stable.
If $\mathcal{C}$ is monotone, then all these are equivalent.

- Recall the collapse for nowhere dense graph classes.
- Vertex coloring and passing to (induced) substructures are in some sense equivalent.
- Non-structure applications: growth rates and wqo

STRUCTURAL CHARACTERIZATIONS

- If $\mathcal{C}$ is monadically NIP, its structures coarsely look like linear orders/are 1-dimensional.
- They have a nice notion of (in)dependence that induces a quasi-order.
- $\mathcal{C}$ is monadically NIP iff (something like) its "linear clique-width" is bounded by a cardinal.

Extremal combinatorics

Question

Can the extra structure from monadic properties be used to improve extremal combinatorial results in hereditary classes?

- Two references:
  Erdős-Hajnal properties for powers of sparse graphs
  Combinatorial and Algorithmic Aspects of Monadic Stability